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Abstract. By adapting Schwartz's treatment of nearly singular potentials we show how dimensional regularization can be used to resolve the continuum eigenvalue difficulty in relativistic bound state problems involving transitional interactions.

Marginally singular potentials, that is attractive potentials which compete with the kinetic energy in the vicinity of the origin, have peculiar difficulties in that simple normalization criteria lead to continuous eigenvalue solutions. (See the review of Frank *et al* (1971), where such potentials are termed transitional.) Only if one makes recourse to more sophisticated concepts (Case 1950) is it possible to select out a discrete set of eigensolutions from among the continuum. The problems associated with such potentials are not just of academic interest since a very practical example which highlights the difficulties is provided by the relativistic bound state equation for a fermion–antifermion composite with a potential derived from a one-particle exchange: owing to the singular character of the kernel, one discovers, for a fixed binding energy and excitation number, a continuous range of permissible coupling constant eigenvalues, contrary to our physical intuition derived from the non-relativistic analogy. In this particular model, say at binding energy equal to twice the fermion mass, two methods have been suggested to pick out a point eigenvalue from the continuum. The first method, proposed by Goldstein (1953) who originally studied this case, was to cut off the relative momentum integrations at some large value Λ and to look for those solutions which did not depend on Λ, or as he stated 'were insensitive to the cutoff procedure'. The second method (Delbourgo *et al* 1967) was to regard the symmetry of the eigenvalue problem as a particular breaking of a problem possessing a higher symmetry. Fortunately, and by no means obviously, both methods agreed on the final answer which corresponds to the eigensolution which is the least singular at the origin.

In this paper we wish to exhibit a third method that is altogether different but which also leads to the same conclusion. It is based on dimensional continuation and makes use of Schwartz's (1976) recent work on nearly singular potentials. Why this method has a chance of success is because the effective potential, continued to a dimension less than 4, is 'safe' relative to the kinetic energy near the origin and thus possesses discrete eigenvalues. We then abstract the answer as the dimension tends to 4. In that sense we differ from Schwartz who keeps away from this limit since he interprets the potential as a physical one which incorporates quantum corrections; furthermore his work is all non-relativistic, though he points out that the technique can be generalized to the relativistic situation.

Let us consider the Bethe–Salpeter equation for a fermion–antifermion wavefunction $\Psi(p, q)$ in a space–time of dimension $4-2w$, where the kernel corresponds to the exchange of a single meson,

$$[(\tfrac{1}{2}p+q)\,.\,\Gamma-m]\Psi(p,q)[(-\tfrac{1}{2}p+q)\,.\,\Gamma-m]$$
$$=g^2\int(\mathrm{d}k/2\pi)^{4-2w}\Gamma_A\Psi(p,k)\Gamma_B\,\Delta^{AB}(k-q). \tag{1}$$

Here p stands for the total and q, k for the relative momenta, while $\Gamma_A\,\Delta^{AB}\Gamma_B$ characterizes the nature of the exchange. Equation (1) is notoriously difficult to solve in the general case so we shall go to the limit $p=0$ where the binding energy is $2m$, and the problem becomes much easier. If x denotes the relative coordinate the equation simplifies to

$$(\mathrm{i}\Gamma\,.\,\vec{\partial}-m)\Psi(x)(\mathrm{i}\Gamma\,.\,\overleftarrow{\partial}-m)=g^2\Gamma_A\Psi(x)\Gamma_B\,\Delta^{AB}(x). \tag{2}$$

In a parity conserving theory $\Delta^{AB}(x)=\delta^{AB}\,\Delta(x)$. Therefore if we expand Ψ into the appropriate complete set† of Γ matrices relevant in that dimension, $\Psi=\Sigma_s\,\Gamma_{[s]}\Psi^{[s]}$; equation (2) reduces to

$$\sum_t \mathrm{Tr}[(\mathrm{i}\Gamma\,.\,\partial-m)\Gamma_{[s]}(\mathrm{i}\Gamma\,.\,\partial-m)\Gamma_{[t]}]\Psi^{[t]}=(4-2w)cg^2\,\Delta(x)\Psi_{[s]} \tag{3}$$

where $c=1$ for scalar or pseudoscalar exchange, $2-2w$ for vector exchange in a Fermi gauge, etc. Equations (3) couple s values which differ by 1. The one exception is the last, pseudoscalar vector (the relativistic analogue of the $^1S_0$ component) which has a clean equation‡ for $\Psi_{[4-2w]}\equiv P$:

$$(\partial^2+m^2)P(x)=cg^2\,\Delta(x)P(x). \tag{4}$$

Only the singularity at $x=0$ is relevant to the continuum eigenvalue difficulty so we shall replace Δ by the massless propagator in (4),

$$\Delta(x)\to D(x)=\Gamma(1-w)(-x^2)^{w-1}/4\pi^{2-w}.$$

Also we shall redefine our eigenvalue g^2 in terms of a dimensionless λ^2 by setting $cg^2\Gamma(1-w)/4\pi^{2-w}=\lambda^2m^{2w}$. Then after rotating to Euclidean space, $r^2=-x^2$, the pseudoscalar equation simplifies to

$$[m^{-2}(\partial/\partial r)^2-1+\lambda^2(m^2r^2)^w]P(r_\nu)=0. \tag{5}$$

Finally we decompose P into a radial function multiplying the appropriate $(4-2w)$-dimensional spherical harmonic§,

$$P(r_\nu)=\sum_N R_N(mr)Y_{N\ldots}(\boldsymbol{r}) \tag{6}$$

to obtain the radial equation‖

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2}+\frac{3-2w}{u}\frac{\mathrm{d}}{\mathrm{d}u}+\frac{\lambda^2}{u^{2-2w}}-1-\frac{N(N+2-2w)}{u^2}\right)R_N(u)=0 \tag{7}$$

† $\Gamma_{[s]}$ stands for the entire set of antisymmetrized, normalized products of s antisymmetric matrices. Thus s runs over the integer values from $0, 1, \ldots, 4-2w$. (See Delbourgo and Prasad (1974) for notational details.)

‡ The other equations either reduce to very similar equations to (4) after contracting over ∂^μ, or else they are pairwise coupled and give essentially the same singularity difficulties as (4).

§ $Y_{N\ldots}(\boldsymbol{r})$ contains a number of degeneracy labels in addition to the Casimir label $N(N+2-2w)$, $N=0, 1, 2, \ldots$. Also $Y_N(\theta)\propto C_N^{1-w}(\cos\theta)$.

‖ We should point out that Gunther (1974) has recently questioned the validity of the Wick rotation in passing from (4) to (7).

where normalization (and perhaps other criteria) must be used to get the eigenvalue λ and eigenfunction R in terms of N for given w.

Initially we fix $w>0$ and eventually we go to the limit as $w\to 0$. Thus we are approaching four dimensions from *below* in order that the potential u^{2w-2} should not dominate over kinetic and centrifugal terms. Before finding out what happens as w decreases to zero let us briefly recall the troubles with continuous λ in four dimensions where $w\equiv 0$. Then (7) reduces to

$$\left(\frac{d^2}{du^2}+\frac{3}{u}\frac{d}{du}+\frac{\lambda^2}{u^2}-1-\frac{N(N+2)}{u^2}\right)R_N(u)=0 \tag{8}$$

with solutions

$$uR_N(u)=AK_\nu(u)+BI_\nu(u); \qquad \nu^2=(N+1)^2-\lambda^2. \tag{9}$$

The normalizability condition, $\int \Psi^*(0,\boldsymbol{x})\Psi(0,\boldsymbol{x})\,d^3\boldsymbol{x}<\infty$, requires us to choose the exponentially damped solution† $K_\nu(u)$, and since in the vicinity of the origin $K_\nu(u)\sim u^{-|\nu|}$,

$$\int R^*R\,d^3\boldsymbol{r}\propto\int r^{-2|\nu|}\,dr.$$

Thus normalizability fixes $\lambda^2-(N+1)^2>-\frac{1}{4}$. On the other hand, reality of ν requires $(N+1)^2\geqslant\lambda^2$. Hence all eigenvalues in the range

$$(N+1)^2-\tfrac{1}{4}<\lambda^2\leqslant(N+1)^2 \tag{10}$$

appear to be admissible, a situation one is loth to accept physically on the basis of all our non-relativistic experience with ordinary Schrödinger equations. The two methods mentioned earlier select the eigenvalue $\lambda=N+1$ and give the corresponding $R_N(u)=K_0(u)/u$ as the least singular solution.

Now let us see how dimensional continuation also leads to the same selection of λ by returning to (7). Following Schwartz (1976), we first rescale our dimensionless variable $u=mr$ (to remove the u dependence connected with λ^2) by putting $u=v^{1/w}$ and then we abstract a threshold factor by setting

$$R=v^{1-1/w}J(v)=u^{w-1}J(u^w). \tag{11}$$

The resulting equation for $J(v)$ is

$$\frac{d^2J}{dv^2}+\frac{1}{v}\frac{dJ}{dv}+\left(\frac{\lambda^2}{w^2}-\frac{(N+1-w)^2}{w^2v^2}\right)J=\frac{J}{w^2v^{2-2/w}} \tag{12}$$

which is recognizable as the equation for a particle in a cylindrically symmetrical potential $V(v)=v^{-2+2/w}/w^2$. As remarked by Schwartz, in the limit of very small w, V approximates to a square well potential: $V\to\infty$ for $v>1$ and $V\to 0$ for $0<v<1$. Therefore we can substitute (12) by the free particle equation, subject to the boundary value $J(v=1)=0$ as $w\to 0$. The solution of (12) in the limit of small w is therefore given by the Bessel function $J_{-1+(N+1)/w}(\lambda v/w)$ after discarding the irregular Y solution. The boundary condition at $v=1$ leads to *discrete* eigenvalues λ_n such that $J_{-1+(N+1)/w}(\lambda_n/w)=0$ where λ_n/w are the zeros of the Bessel function. One can ascertain from the properties of Bessel functions (Abramowitz and Stegun 1968) that

† The sign of the root for ν is irrelevant since $K_\nu=K_{-\nu}$.

for fixed z and asymptotically large μ that $J_\mu(\mu z)$ has zeros located in the neighbourhood of $z=1$. The position of these zeros can then be found by means of the asymptotic behaviour

$$J_\mu(\mu+z\mu^{\frac{1}{3}})=(2/\mu)^{\frac{1}{3}}A\mathrm{i}(-2^{\frac{1}{3}}z)+\mathrm{O}(1/\mu)$$

upon making the identifications

$$\mu=-1+(N+1)/w \qquad \text{and} \qquad \lambda=N+1+z(N+1)^{\frac{1}{3}}w^{\frac{2}{3}}+\mathrm{O}(w).$$

The zeros of the Airy function are tabulated in the standard references (see Abramowitz and Stegun 1968) and occur at

$$2^{\frac{1}{3}}z_0=2\cdot34, \qquad 2^{\frac{1}{3}}z_1=4\cdot08, \qquad 2^{\frac{1}{3}}z_2=5\cdot52, \ldots. \tag{13}$$

Thus one finishes up with the discrete set of eigenvalues

$$\lambda_n=(N+1)+z_n(N+1)^{\frac{1}{3}}w^{\frac{2}{3}}+\ldots \tag{14}$$

all of which crowd in to $(N+1)$ for fixed n, as $w\to0$. The corresponding eigenfunctions tend to

$$R_N(u)=u^{w-1}J_{(N+1)/w}[(N+1)u^w/w] \tag{15}$$

in the limit of small w. These must reduce to $K_0(u)/u$ as we know that the ancestral equation (7) collapses into the four-dimensional form (8) for $w=0$. However we have been unable to discover the relevant formula from the texts on Bessel functions which clearly demonstrates this equivalence as $w\to0$.

The one weak point in our analysis is the question of non-uniform limits whereby we have fixed n at a finite integer value in (14) before we pass to $w=0$. However since specific orders of limits have been adopted in past applications of dimensional regularization (for instance in treating infrared divergences, or else to prove the vanishing of tadpole graphs) our attitude to this question is no different from that of previous authors: namely, that of maintaining consistency, which we believe we have done†.

Evidently this technique of dimensional continuation can be applied to other relativistic problems involving higher spin particles that are characterized by transitional interactions.

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† If one chooses another order of limits, whereby $n\to\infty$ simultaneously with $w\to0$ one would arrive at unbounded values for λ in (14). And because the number of nodes becomes infinitely large, the answer would depend sensitively on the shape of the potential near $v=1$; i.e. the results would depend on the *way* that one approaches four dimensions.